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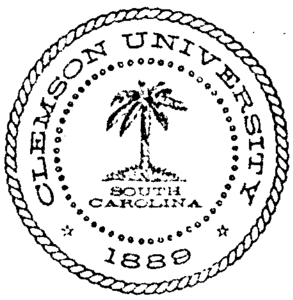

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STATISTICAL INFERENCE IN A NEW BIVARIATE FAILURE MODEL

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ABSTRACT

In a previous paper a new bivariate failure model was introduced and its properties investigated. In this paper we consider estimation of the parameters of this model in a special case, for both complete and incomplete samples.

1. Introduction. Many biv riate exponential distributions have been suggested in the literature. These distributions differ in the various properties that describe them. One of the properties that it is natural to demand of such a distribution is that it possess exponential marginals, and many of the proposed distributions have this property including those suggested by Gumbel [6], Downton [4], Hawkes [7], and Marshall and Olkin [8].

Another highly desirable property is the lack of memory property. In the univariate case this property is

$$\Pr\{S>s+t\} = \Pr\{S>s\} \Pr\{T>t\}$$

for all s, t > 0 and is enjoyed only by the exponential distribution. The natural extension of this property to two dimensions is that

$$P_r\{S>s_1+t_1,\ T>s_2+t_2\} \quad \Pr\{S>s_1,\ T>s_2\} \Pr\{S>t_1,\ T>t_2\}$$

for all s_1, t_1, $s_2, t_2 \geq 0$. However, this definition is too restrictive to be useful. Marshall and Olkin [8] have shown that the joint distribution of two independent exponential distributions is the only distribution that has this property. Consequently, they define the bivariate lack of memory property (LMP) as $\Pr\{S>\Delta+s,\ T>\Delta+t\}$

$=\Pr\{S>\Delta, T>\Delta\} \Pr\{S>s,\ T>t\}$ for all s, t, $\Delta \geq 0$. Unfortunately, this definition does not yield a unique distribution. In addition to Marshall and Olkin [8], distributions that have this property include those of Freund [5] and Block and Basu [1].

The model proposed by Marshall and Olkin [8] has both exponential marginals and the LMP and is the most widely referenced bivariate failure model. Their survival distribution is

$$\Pr\{S>s,\ T>t\} = \exp(-\lambda_1 s-\lambda_2 t-\lambda_{12}\ \max\ (s,t))\ \text{for}$$

$s,t \geq 0$ and fixed parameters $\lambda_1,\lambda_2,\lambda_{12} > 0$. They show that this distribution is the only one with both of the properties. Note that in this model $\Pr\{S=T\} = \lambda_{12}/(\lambda_1+\lambda_2+\lambda_{12})>0$, so that there is a singular component in the distribution. Computationaly this poses some difficulties, but these can be overcome. Furthermore, this model can handle the simultaneous failure of both components. Also, in this model,

$$\Pr\{S>s+\Delta|S>s,\ T>s\} = \Pr\{S>s+\Delta|S>s,\ T\leq s\}$$

which implies that conditioned on the fact that one component if functioning at time s, the distribution of its residual lifetime is independent of whether or not the other component has failed. Freund [5] has derived a bivariate failure distribution using the assumption that at the failure of one component the distribution of the residual lifetime of the other component is changed. The marginal distributions in Freund's model are not exponential but are mixtures of exponentials. In a previous paper [9] Freund's model was generalized and some of the properties of the new bivariate failure model were investigated. Here we consider a special case of the new model with special reference to estimating its parameters. The model is defined in section 2 and the first two moments are

calculated. In section 3, the survival distribution and the LMP are discussed.

Maximum likelihood estimation is taken up in section 4. Explicit expressions for the estimators are given. Properties of these estimators are derived in section 5 including their joint asymptotic distribution. Finally, in section 6, maximum likelihood estimation is discussed for an incomplete sample.

2. Model Definition and Moments. Let A and B be two components of a system with lifetimes S and T respectively. For given random variables X, Y, U, V we write

$$S = \begin{cases} X & \text{if} \quad X < Y \\ Y + U & \text{if} \quad X \geq Y \end{cases}, \quad T = \begin{cases} X + V & \text{if} \quad X < Y \\ Y & \text{if} \quad X \leq Y \end{cases} \tag{2.1}$$

Here we take X, Y, U, V to be mutually independent with X distributed as exponential with parameter α, Y distributed as exponential with parameter β. Let

$$\Pr\{U>t\} = qe^{-\alpha' t} \quad \text{and} \quad \Pr\{V>t\} = qe^{-\beta' t},$$

where $\alpha,\beta,\alpha',\beta'>0$ and $0\leq q\leq 1$. Freund's model corresponds to the case when $q = 1$. The parameter q allows for simultaneous failure of the components, since $\Pr\{S = T\} = 1-q = p$.

Tosch and Holmes [9] have discussed this model without distribution assumptions and have derived (among other things) the Laplace-Stieltjes (L-S) transform of the joint distribution of S and T. We now state a result from that paper that will be used in the present context.

Lemma 1:

If X ~ exp (α) and Y ~ exp(β), then.

i) $f^*(a,b) = \frac{1}{a+b+\alpha+\beta}[\alpha f_V^*(b) + \beta f_U^*(a)]$, where $f^*(a,b)$ is the transform of the joint distribution.

ii) $E(S) = \frac{1}{\alpha+\beta}[1+\beta E(U)]$.

iii) $Var(S) = \frac{1}{(\alpha+\beta^2)}[1+\beta^2 Var(U) + \alpha\beta E(U^2)]$,

iv) $Cov(S,T) = \frac{1}{(\alpha+\beta)^2}[1-\alpha\beta E(U)E(V)]$.

In this specific case $f_U^*(a) = p+\frac{\alpha' q}{a+\alpha}$, $f_V^*(b) = p + \frac{\beta' q}{b+\beta}$, $E(U) = \frac{q}{\alpha'}$, $E(U^2) = \frac{2q}{(\alpha')^2}$ and $Var(U) = \frac{q}{(\alpha')^2}$. By substituting these quantities into Lemma 1 we see that

Theorem 2: For the given model

i) $f^*(a,b) = \frac{1}{\alpha+\beta+a+b}[p(\alpha+\beta) + q(\frac{\alpha\beta'}{b+\beta'} + \frac{\beta\alpha'}{a+\alpha'})]$,

ii) $E(S) = \frac{\alpha'+\beta q}{\alpha'(\alpha+\beta)}$,

iii) $Var(S) = \frac{1}{(\alpha')^2(\alpha+\beta)^2}[(\alpha')^2 + q\beta(2-q\beta+2\alpha)]$,

iv) $Cov(S,T) = \frac{1}{\alpha'\beta'(\alpha+\beta)^2}[\alpha'\beta'-q^2\alpha\beta]$.

The moments of T follow similarly. As in Freund's model the correlation, $\rho(S,T)$, is seen to vary between $-\frac{1}{3}$ and 1.

3. Survival Distribution and the Lack of Memory Property.

To arrive at the survival distribution we again call upon a result for the general case.

Lemma 3:

$$\overline{F}(s,t) = \begin{cases} \overline{F}_X(t)\overline{F}_Y(t) + \int_s^t \overline{F}_V(t-x)\overline{F}_Y(x)\,dF_X(x) & \text{if } s<t \\ \overline{F}_X(s)\overline{F}_Y(s) & \text{if } s=t \\ \overline{F}_X(s)\overline{F}_Y(s) + \int_t^s \overline{F}_U(s-y)\overline{F}_X(y)\,dF_Y(y) & \text{if } s>t \end{cases}$$

Thus for s>t we have

$$\begin{aligned} \overline{F}(s,t) &= e^{-(\alpha+\beta)s} + \int_t^s qe^{-\alpha'(s-y)-\alpha y}\beta e^{-\beta y}dy \\ &= e^{-(\alpha+\beta)s} + q\beta e^{-\alpha' s}\int_t^s e^{-(\alpha+\beta-\alpha')y}dy \\ &= e^{-(\alpha+\beta)s} + \frac{q\beta e^{-\alpha' s}}{\alpha+\beta-\alpha'}\left[e^{-(\alpha+\beta-\alpha')t} - e^{-(\alpha+\beta-\alpha')s}\right] \qquad (3.1) \end{aligned}$$

The last equation assumes that $\alpha+\beta-\alpha' \neq 0$. If $\alpha+\beta-\alpha' = 0$ then (3.1) becomes

$$\overline{F}(s,t) = e^{-(\alpha+\beta)s} + q\beta e^{-\alpha' s}(s-t), \text{ if } s>t. \qquad (3.2)$$

For the remainder of the paper, it will be assumed that $\alpha+\beta-\alpha' \neq 0$ and also that $\alpha+\beta-\beta' \neq 0$. The calculations are similar when s<t, so that

Theorem 4:

$$\overline{F}(s,t) = \begin{cases} e^{-(\alpha+\beta)t} + \frac{q\alpha e^{-\beta' t}}{\alpha+\beta-\beta'}\left[e^{-(\alpha+\beta-\beta')s} - e^{-(\alpha+\beta-\beta')t}\right] & \text{if } s<t, \\ e^{-(\alpha+\beta)s} & \text{if } s=t, \\ e^{-(\alpha+\beta)s} + \frac{q\beta e^{-\alpha' s}}{\alpha+\beta-\alpha'}\left[e^{-(\alpha+\beta-\alpha')t} - e^{-(\alpha+\beta-\alpha')s}\right] & \text{if } s>t. \end{cases}$$

The marginals are given by

Corollary 5:

$$\text{i)}\quad \bar{F}_S(s) = \frac{\alpha+p\beta-\alpha'}{\alpha+\beta-\alpha'}\, e^{-(\alpha+\beta)s} + \frac{q\beta}{\alpha+\beta-\alpha'}\, e^{-\alpha' s} \quad ,$$

$$\text{ii)}\quad \bar{F}_T(t) = \frac{\alpha+\beta p\alpha-\beta'}{\alpha+\beta-\beta'}\, e^{-(\alpha+\beta)t} + \frac{q\alpha}{\alpha+\beta-\beta'}\, e^{-\beta' t} \quad ,$$

We see that the marginals are mixtures of exponentials.

Theorem 6: The survival distribution given in Theorem 4 has the LMP.

Proof: We must show that $\bar{F}(s+\Delta,\ t+\Delta) = \bar{F}(s,t)\bar{F}(\Delta,\Delta)$ for all $s,\ t,\Delta \geq 0$.

If $s=t$ then $s+\Delta = t+\Delta$ and $\bar{F}(s+\Delta,\ s+\Delta) = e^{-(\alpha+\beta)(s+\Delta)}$

$$= e^{-(\alpha+\beta)s}e^{-(\alpha+\beta)\Delta} = \bar{F}(s,s)\bar{F}(\Delta,\Delta).$$

If $s<t$ then $s+\Delta < t+\Delta$ so that

$$\bar{F}(s+\Delta,t+\Delta) = e^{-(\alpha+\beta)(t+\Delta)} + \frac{q\alpha e^{-\beta'(t+\Delta)}}{\alpha+\beta-\beta'}\,[e^{-(\alpha+\beta-\beta')(s+\Delta)} - e^{-(\alpha+\beta-\beta')(t+\Delta)}]$$

$$= e^{-(\alpha+\beta b)\Delta}[e^{-(\alpha+\beta)t} + \frac{q\alpha e^{-\beta'(t+\Delta)}}{\alpha+\beta-\beta'}\,(e^{-(\alpha+\beta-\beta')s}e^{\beta'\Delta} - e^{-(\alpha+\beta-\beta')t}e^{\beta'\Delta})]$$

$$= e^{-(\alpha+\beta)\Delta}[e^{-(\alpha+\beta)t} + \frac{q\alpha e^{-\beta' t}}{\alpha+\beta-\beta'}\,(e^{-(\alpha+\beta-\beta')s} - e^{-(\alpha+\beta-\beta')t})]$$

$= \bar{F}(\Delta,\Delta)\bar{F}(s,t)$. The result follows similarly for $s>t$. QED

4. Maximum Likelihood Estimation. The measure determined by the survival function is not absolutely continuous with respect to μ_2, Lebesgue measure on R_2. Therefore, there does not exist a probability density function (pdf) for (S,T) with

respect to μ_2. Bhattacharrya and Johnson [2], when dealing with estimation in the Marshall-Olkin model, overcame this problem by considering the following measure on R_2^+, where $R_2^+ = \{(x,y) \mid x>0,\ y>0\}$.

Let

$$\mu(A) = \mu_2(A) + \mu_1(\{x \mid x>0,\ (x,x)\varepsilon A\}), \qquad (4.1)$$

where μ_1 is Lebesgue measure on R_1. This measure will suffice for our purpose here also. The measure μ is σ-finite on R_2^+ and the measure determined by the survival function is absolutely continuous with respect to μ. It can be shown that

<u>Theorem 7</u>: The pdf of (S,T) with respect to μ is given by

$$f(s,t) = \begin{cases} \alpha\beta' q e^{-(\alpha+\quad ')s-\beta' t} & \text{if } s<t, \\ p(\alpha+\beta)e^{-(\alpha+\beta)s} & \text{if } s=t, \\ \beta\alpha' q e^{-(\alpha+\beta-\alpha')t-\alpha' s} & \text{if } s>t. \end{cases}$$

We are now in a position to write down the likelihood function with respect to μ. Consider a sample of N observations $\{(s_1,t_1),(s_2 t_2),\ldots,(s_N,t_N)\}$. The likelihood function is $L = \prod_{i=1}^{N} f(s_i,t_i)$. The following notation will simplify the expression.

Let

$$N_1 = \#\{(s_i,t_i) \mid s_i<t_i\},$$

$$N_2 = \#\{(s_i,t_i) \mid s_i>t_i\},$$

$$N_3 = \#\{(s_i,t_i) \mid s_i = t_i\},$$

$$S_1 = \sum_{s_i<t_i} s_i,$$

$$S_2 = \sum_{s_i>t_i} s_i,$$

$$R = \sum_{s_i = t_i} s_i,$$

$$T_1 = \sum_{s_i<t_i} t_i$$

$$T_2 = \sum_{s_i>t_i} t_i,$$ where #A is the number of

items in the set A. In this notation

$$L = (\alpha\beta')^{N_1}(1-p)^{N_1+N_2}p^{N_3}(\alpha+\beta)^{N_3}(\alpha\beta')^{N_2}\exp[-(\alpha+\beta)(S_1+R+T_2)-\alpha'(S_2-T_2)$$

$$-\beta'(T_1-S_1)]. \quad (4.2)$$

Let $\ell = \ell_n L$ be the log likelihood function. Then

$$\ell = N_1\ell_n(\alpha) + N_1\ell_n(\beta') + (N_1+N_2)\ell_n(1-p) + N_3\ell n(p) + N_3\ell n(\alpha+\beta)$$

$$+ N_2\ell_n(\beta) + N_2\ell n(\alpha')-(\alpha+\beta)(S_1+R+T_2-\alpha'(S_2-T_2)-\beta'(T_1-S_1) \quad (4.3$$

Let $\underset{\sim}{\theta} = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = (\alpha,\beta,\alpha',\beta',p)$ to be the vector of unknown parameters. The likelihood equations are then

$$N_1/\alpha + N_3/(\alpha+\beta) - (S_1+R+T_2) = 0,$$

$$N_2/\beta + N_3/(\alpha+\beta) - (S_1+R+T_2) = 0,$$

$$N_2/\alpha' - (S_2-T_2) = 0, \qquad (4.4)$$

$$N_1/\beta' - (T_1-S_1) = 0,$$

$$-(N_1+N_2)/(1-p) + N/p = 0.$$

By solving these equations we can obtain the maximum likelihood estimates.

<u>Theorem 8</u>:

i) If $N_1 = N_2 = 0$ (so that $N_3 = N$), then $\hat{p} = 1$ and α, β, α', β' cannot be estimated.

ii) If $N_1 = 0$, but $N_2 \neq 0$, then $\hat{p} = N_3/N$, $\hat{\alpha} = 0$, $\hat{\beta} = \frac{N}{S_1+R+T_2}$,

$\hat{\alpha}' = N_2/(S_2-T_2)_1$ and β' cannot be estimated.

iii) If $N_1 \neq 0$, but $N_2 = 0$, then $\hat{p} = N_3/N$, $\hat{\beta} = 0$, $\hat{\alpha} = \frac{N}{S_1+R+T_2}$,

$\hat{\beta}' = N_1/(T_1-S_1)$, and α' cannot be estimated.

iv) If $N_1 \neq 0$ and $N_2 \neq 0$, then

$$\hat{\alpha} = \frac{N}{S_1+R+T_2}\left(\frac{N_1}{N_1+N_2}\right),$$

$$\hat{\beta} = \frac{N}{S_1+R+T_2}\left(\frac{N_2}{N_1+N_2}\right),$$

$$\hat{\alpha}' = \frac{N_2}{S_2-T_2},$$

$$\hat{\beta}' = \frac{N_1}{T_1-S_1},$$

$$\hat{p} = \frac{N_3}{N}.$$

Proof of iv): The parameter space is $\Omega = \{(\alpha,\beta,\alpha',\beta',p) \mid \alpha>0,\beta>0,\alpha'>0,\beta'>0,\ 0\leq p\leq 1\}$. On the boundary of Ω, $L = 0$. All of the partial derivatives of L exist and are continuous on the interior of Ω. Finally $L > 0$ on the interior of Ω given that $N_1 \neq 0$, $N_2 \neq 0$. Therefore since $(\hat{\alpha},\hat{\beta},\hat{\alpha}',\hat{\beta}',\hat{p})$ is the unique point where the gradient vanishes, L must attain its maximum at that point. Q.E.D.

5. Properties of the Maximum Likelihood Estimates

Theorem 9:

i) $E(\hat{\alpha}) = (N/(N-1))\alpha$,

ii) $E(\hat{\beta}) = (N/(N-1))\beta$,

iii) $E(\hat{p}) = p$.

Proof: First the following two observations are made: (N_1,N_2,N_3) has a trinomial distribution with parameters

$$(N,\ \frac{q\alpha}{\alpha+\beta},\ \frac{q\beta}{\alpha+\beta},\ P); \tag{5.1}$$

$(N_1 | N_3$ has a binomial distribution with parameters

$$(N-N_3,\ \frac{\alpha}{\alpha+\beta})\ . \tag{5.2}$$

i) $$E(\frac{N_1}{N_1+N_2}) = E(E(\frac{N_1}{N_1+N_2}|N_3)) = E(E(\frac{N_1}{N-N_3}|N_3)) = E(\frac{1}{N-N_3}E(N_1|N_3)).$$

$$= E(\frac{1}{N-N_3}(N-N_3)\frac{\alpha}{\alpha+\beta}) \quad \text{by (5.2)}$$

$$= \frac{\alpha}{\alpha+\beta}\ .$$

Now S_1+R+T_2 can be written as $\sum_{i=1}^{N} \min(s_i,t_i)$, but $\min(s_i,t_i)$ has the same distribution as $\min(X,Y) \sim \exp(\alpha+\beta)$, so that S_1+R+T_2 has an Erlang distribution of order N and parameter $\alpha+\beta$. Therefore

$$E(\frac{N}{S_1+R+T_2}) = N\int_0^\infty \frac{1}{x} \frac{(\alpha+\beta)^N x^{N-1}}{(N-1)!} e^{-(\alpha+\beta)x} dx = (\frac{N}{N-1})(\alpha+\beta).$$

Finally S_1+R+T_2 is independent of $\frac{N_1}{N_1+N_2}$ so that

$$E(\hat{\alpha}) = E[\frac{N}{S_1+R+T_2}(\frac{N_1}{N_1+N_2})] = E(\frac{N}{S_1+R+T_2})E(\frac{N_1}{N_1+N_2}) \text{ by independence}$$

$$= \frac{N}{N-1}(\alpha+\beta)\frac{\alpha}{\alpha+\beta} = (\frac{N}{N-1})\alpha .$$

ii) $E(\hat{\beta})$ follows similarly,

iii) $E(\hat{p}) = E(\frac{N_3}{N} = \frac{p^N}{N}$ by (5.1)

$= p.$ Q.E.D.

In like fashion it can be shown that

<u>Theorem 10</u>:

i) $E(\hat{\alpha}'|N_2) = (\frac{N_2}{N_2-1})\alpha'$ if $N_2>1$,

ii) $E(\hat{\beta}'|N_1) = (\frac{N_1}{N_1-1})\beta'$ if $N_1>1$.

Using a Lehmann, Scheffè partioning operation (c.f. Zacks [10], p. 50) it can be shown that

<u>Theorem 11</u>: The vector $(N_1, N_2, S_1+R+T_2, S_2-T_2T_1-S_1)$ is a minimal sufficient statistic of the sample $\{s_1,t_1),\ldots,(S_N,t_N)\}$.

From this we see that the vector of maximum likelihood estimates is also a minimal sufficient statistic.

To investigate the asymptotic properties of the maximum likelihood estimate, we will use the conditions presented by Chanda [3, p. 56]. Of the three conditions, the first two are the usual Cramer-Rao regularity conditions. To establish these results, we need to choose a 5-dimensional interval Ω^*, which contains the true values of the parameter.

Let

$$\Omega^* = \{\underset{\sim}{\theta} \mid 0<\varepsilon_i<\theta_i<M_i<\infty,\ i = 1,2,3,4,5\}$$

for some pre-chosen constants ε_i, M_i, $i = 1, 2, 3, 4, 5$. Here $M_5<1$. For instance, if $\varepsilon_1 = 10^{-9}$ and $M_1 = 10^{10}$, we restrict the mean of X to $(10^{-10}, 10^{9})$. From physical considerations, we can certainly arrive at the necessary bounds. In Ω^*, it is straightforward to obtain the necessary dominating functions. The third condition requires the positive definiteness of the information matrix. We begin by finding the Hessian matrix Q. Let $Q = (q_{ij})$, where $q_{ij} = \frac{\alpha^2 \ell}{\alpha\theta_i \alpha\theta_j}$, ℓ is the log likelihood function (4.3) and $\underset{\sim}{\theta} = (\alpha,\beta,\alpha',\beta', p)$. By direct calculation

$$-Q = \begin{bmatrix} \frac{N_1}{\alpha^2} + \frac{N_3}{(\alpha+\beta)^2} & \frac{N_3}{(\alpha+\beta)^2} & 0 & 0 & 0 \\ \frac{N_3}{(\alpha+\beta)^2} & \frac{N_2}{\beta^2} + \frac{N_3}{(\alpha+\beta)^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{N_2}{(\alpha')^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{N_1}{(\alpha')^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{N_1+N_2}{(1-p)^2} + \frac{N_3}{p^2} \end{bmatrix} \tag{5.3}$$

Note that in Ω^*, $0<p<1$. The information matrix $\Sigma^{-1} = E(-N^{-1}Q)$ can now be evaluated. Since $E(N_1) = (\frac{\alpha}{\alpha+\beta})N, E(N_2) = (\frac{q\beta}{\alpha+\beta})N$, and $E(N_3) = p^N$, we have

$$\Sigma^{-1} = \begin{bmatrix} \frac{\alpha+\beta-p\beta}{\alpha(\alpha+\beta)^2} & \frac{p}{(\alpha+\beta)^2} & 0 & 0 & 0 \\ \frac{p}{(\alpha+\beta)^2} & \frac{\alpha+\beta-p\alpha}{\beta(\alpha+\beta)^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-p)\beta}{(\alpha')^2(\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-p)\alpha}{(\beta')^2(\alpha+\beta)} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{p(1-p)} \end{bmatrix}$$

This is easily seen to be positive definite. Thus all of thc conditions are met and we have

Theorem 12:

$\hat{\underset{\sim}{\theta}}_N \to \underset{\sim}{\theta}$ as $N\to\infty$ with probability one.

Theorem 13:

$N^{1/2}(\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta})$ is asymptotically distributed as multivariate normal with mean $\underset{\sim}{0}$ and convariance matrix Σ.

6. Estimation with an Incomplete Sample. Suppose that we have an incomplete sample, that is, some of the components do not fail in the alotted time. Again let N be the total number of samples taken and let N_1, N_2, N_3, S_1, S_2, R, T_1, T_2 be as before. Let s_i be the time when component A failed or the time that the $i^{\underline{th}}$ experiment was stopped, if A did not fail. Similarly for t_i, Let

M_1 = #{(s_i,t_i)|A failed by B did not},

M_2 = #{(s_i,t_i)|A did not fail but B did},

M_3 = # (s_i,t_i)|neither A nor B failed},

S_1' = sum of s_i's when A failed but B did not,

S_2' = sum of s_i's when A did not fail but B did,

T_1' = sum of t_i's when A failed but B did not,

T_2' = sum of t_i's when A did not fail but B did,

R' = sum of s_i's when neither A nor B failed.

We want to obtain the likelihood elements for these incomplete samples. The following is derived directly from the survival

function.

Lemma 14:

i) for $s<t$, $\Pr\{S=s,\ T>t\} = -\dfrac{\partial \bar{F}(s,t)}{\partial s} = q\alpha e^{-\beta' t-(\alpha+\beta-\beta')s}ds$

ii) for $s = t$, $\Pr\{S>s,\ T>t\} = e^{-(\alpha+\beta)s}$

iii) for $s>t$, $\Pr\{S>s,\ t=t\} = q\beta e^{-\alpha' s-(\alpha+\beta-\alpha')t}dt.$

The likelihood function, L, can now be obtained.

$$L = \alpha^{N_1+M_1}(\beta')^{N_1}(1-p)^{N_1+N_2+M_1+M_2}p^{N_3}(\alpha+\beta)^{N_3}\beta^{N_2+M_2}(\alpha')^{N_2}$$

$$\cdot\exp[-(\alpha+\beta)(S_1+R+T_2+S_1'+R'+T_2')-\alpha'(S_2-T_2+S_2'-T_2')-\beta'(T_1-S_1+T_1'-S_1')]. \quad (6.1)$$

If we let $\ell = \ell_n L$ be the log likelihood function, the likelihood equations are given by $\dfrac{\partial \ell}{\partial \theta_i} = 0$, $i = 1,2,3,4,5$. Again the likelihood function is zero on the boundary and positive on the interior of the parameter space. Therefore the unique solution to the likelihood equations is the maximum likelihood estimate. Excluding the cases where some of the parameters are not estimable, we have

Theorem 15: The maximum likelihood estimates in the incomplete sample are given by:

$$\hat{\alpha} = \frac{N_1+M_1+N_2+M_2+N_3}{S_1+R+T_2+S_1'+R'+T_2'}\left[\frac{M_1+M_2}{N_1+M_1+N_2+M_2}\right],$$

$$\hat{\beta} = \frac{N_1+M_1+N_2+M_2+N_3}{S_1+R+T_2+S_1'+R'+T_2'}\left[\frac{N_2+M_2}{N_1+M_1+N_2+M_2}\right],$$

$$\hat{\alpha} = N_2/(S_2-T_2+S_2'-T_2'),$$

$$\hat{\beta}' = N_1/(T_1-S_1+T_1'-S_1'),$$

$$\hat{p} = N_3/(N_1+N_2+M_1+M_2+N_3).$$

7. Discussion. The model presented differs from Freund's model in the addition of another parameter to include the possibility of simultaneous failure of both components. In biological applications, this could be the simultaneous loss of paired organs by some catastrophe or disease.

In Marshall and Olkin's model, the residual lifetime of one component is independent of whether or not the other component has failed. This is not true of the proposed model. In many applications, the failure of one component puts more (possibly less) strain on the remaining one, i.e., when one kidney fails. Lastly, Marshall and Olkin's model allows only for positive correlation in the component lifetimes, while the proposed model allows for some negative correlation as well.

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In a previous paper a new general bivariate failure model was introduced and its properties investigated. Here we consider the problem of parameter estimation in the exponential case for both complete and incomplete samples.